3D Kalman Track Fit Using Space Points and Hits as Measurements

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Contents

1 Introduction

This note contains detailed examples of the Kalman fit algorithm using either space points or hits (larsoft class `RecoBase/Hit`) as measurements. In either case, we assume that we want to use a measurement (space point or hit) to improve a track hypothesis specified on surface of constant z.

2 Track Hypothesis

We use the following track parameters to describe a track on a z-plane surface: (x, $x' = dx/dz$, y, $y' = dy/dz$, $1/p$). Formally, we define a state vector $\mathbf{x}$ as consisting of the following five track parameters.

$$\mathbf{x} = \begin{bmatrix} x \\ x' \\ y \\ y' \\ 1/p \end{bmatrix}. \tag{1}$$

The track parameter state vector has an error matrix C, which for simplicity we will assume initially to have a 2+2+1 block diagonal structure.

$$\mathsf{C} = \begin{bmatrix} \sigma_x^2 & \sigma_{xx'}^2 & 0 & 0 & 0 \\ \sigma_{xx'}^2 & \sigma_{x'}^2 & 0 & 0 & 0 \\ 0 & 0 & \sigma_y^2 & \sigma_{yy'}^2 & 0 \\ 0 & 0 & \sigma_{yy'}^2 & \sigma_{y'}^2 & 0 \\ 0 & 0 & 0 & 0 & \sigma_{p^{-1}}^2 \end{bmatrix}. \tag{2}$$

That is, we assume that the errors in the two Cartesian views are not correlated, and that the momentum error is not correlated with any other error (consistent with the absence of a magnetic field). Also, we can be virtually certain that the position and slope errors are positively correlated ($\sigma_{xx'}^2 > 0$ and $\sigma_{yy'}^2 > 0$). For example, if the errors are entirely due to multiple Coulomb scattering over a distance L with rms scattering angle σ_θ, then (in the small angle approximation),

$$\sigma_x^2 = \frac{L^2}{3}\sigma_\theta^2, \tag{3}$$

$$\sigma_{xx'}^2 = \frac{L}{2}\sigma_\theta^2, \tag{4}$$

$$\sigma_{x'}^2 = \sigma_\theta^2, \tag{5}$$

with correlation coefficient $\rho_x = \sigma_{xx'}^2/(\sigma_x\sigma_{x'}) = \sqrt{3}/2$.

3 Space Point as Measurement

Given a measured space point (x_m, y_m, z_m), the z-coordinate defines the measurement surface and the track surface. We define a two-dimensinoal measurement vector $\mathbf{m}$ consisting of the x and y coordinates.

$$\mathbf{m} = \begin{bmatrix} x_m \\ y_m \end{bmatrix}. \tag{6}$$

The measurement vector has an error matrix V which can be obtained by projecting the space point error matrix onto the measurement surface. We write the measurement error matrix as follows.

$$\mathsf{V} = \begin{bmatrix} \sigma_{mx}^2 & 0 \\ 0 & \sigma_{my}^2 \end{bmatrix}. \tag{7}$$

Again, we assume x and y are uncorrelated.

We can write the predicted measurement vector as

$$\mathbf{m}_{\text{pred}} = \begin{bmatrix} x \\ y \end{bmatrix}, \tag{8}$$

where x and y are simply the corresponding track parameters. The error matrix of the prediction in measurement coordinates is denoted by T, and is related to track parameter error matrix C as

$$\mathsf{T} = \mathsf{HCH}^T, \tag{9}$$

where the matrix H is the Jacobian of the prediction function. In the case of space points,

$$\begin{aligned}
\mathsf{H} &= \frac{\partial\,(x,y)}{\partial\,(x,x',y,y',p^{-1})}, && (10)\\
&= \begin{bmatrix} \frac{\partial x}{\partial x} & \frac{\partial x}{\partial x'} & \frac{\partial x}{\partial y} & \frac{\partial x}{\partial y'} & \frac{\partial x}{\partial (p^{-1})} \\ \frac{\partial y}{\partial x} & \frac{\partial y}{\partial x'} & \frac{\partial y}{\partial y} & \frac{\partial y}{\partial y'} & \frac{\partial y}{\partial (p^{-1})} \end{bmatrix}, && (11)\\
&= \begin{bmatrix} 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \end{bmatrix}. && (12)
\end{aligned}$$

Finally, we define the residual vector $\mathbf{r}$ is the difference between the prediction and measurement.

$$\mathbf{r} = \mathbf{m} - \mathbf{m}_{\text{pred}} = \begin{bmatrix} x_m - x \\ y_m - y \end{bmatrix}, \qquad (13)$$

The residual vector has its own error matrix R, which includes contributions from the measurement and track.

$$\begin{aligned}
\mathsf{R} &= \mathsf{V} + \mathsf{T}, && (14)\\
&= \begin{bmatrix} \sigma_{mx}^2 + \sigma_x^2 & 0 \\ 0 & \sigma_{my}^2 + \sigma_y^2 \end{bmatrix}. && (15)
\end{aligned}$$

The Kalman algorithm uses an updating formula such that the updated state vector $\mathbf{x}'$ is deviated compared to the original state vector $\mathbf{x}$ by an amount that is proportional to the residual vector $\mathbf{r}$.

$$\mathbf{x}' = \mathbf{x} + \mathsf{K}\mathbf{r}, \qquad (16)$$

where K is called the Kalman gain matrix. The gain matrix can be calculated in terms of previously defined quantities as

$$\begin{aligned}
\mathsf{K} &= \mathsf{C}\mathsf{H}^T\mathsf{R}^{-1}, && (17)\\
&= \begin{bmatrix} \sigma_x^2 & \sigma_{xx'}^2 & 0 & 0 & 0 \\ \sigma_{xx'}^2 & \sigma_{x'}^2 & 0 & 0 & 0 \\ 0 & 0 & \sigma_y^2 & \sigma_{yy'}^2 & 0 \\ 0 & 0 & \sigma_{yy'}^2 & \sigma_{y'}^2 & 0 \\ 0 & 0 & 0 & 0 & \sigma_{p^{-1}}^2 \end{bmatrix} \begin{bmatrix} 1 & 0 \\ 0 & 0 \\ 0 & 1 \\ 0 & 0 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} \frac{1}{\sigma_{mx}^2+\sigma_x^2} & 0 \\ 0 & \frac{1}{\sigma_{my}^2+\sigma_y^2} \end{bmatrix}, && (18)\\
&= \begin{bmatrix} \frac{\sigma_x^2}{\sigma_{mx}^2+\sigma_x^2} & 0 \\ \frac{\sigma_{xx'}^2}{\sigma_{mx}^2+\sigma_x^2} & 0 \\ 0 & \frac{\sigma_y^2}{\sigma_{my}^2+\sigma_y^2} \\ 0 & \frac{\sigma_{yy'}^2}{\sigma_{my}^2+\sigma_y^2} \\ 0 & 0 \end{bmatrix}. && (19)
\end{aligned}$$

The updated track error matrix can be calculated in terms of the Kalman gain matrix using the following formula.

$$\mathsf{C}' = (1 - \mathsf{K}\mathsf{H})\,\mathsf{C}. \qquad (20)$$

Using the Kalman updating formula, the filtered track parameter state vector is

$$\mathbf{x}' = \begin{bmatrix} \frac{x\sigma_{mx}^2 + x_m\sigma_x^2}{\sigma_{mx}^2+\sigma_x^2} \\ x' + \left(\frac{\sigma_{xx'}^2}{\sigma_{mx}^2+\sigma_x^2}\right)(x_m - x) \\ \frac{y\sigma_{my}^2 + y_m\sigma_y^2}{\sigma_{my}^2+\sigma_y^2} \\ y' + \left(\frac{\sigma_{yy'}^2}{\sigma_{my}^2+\sigma_y^2}\right)(y_m - y) \\ 1/p \end{bmatrix}. \tag{21}$$

The filtered track error matrix is

$$\mathsf{C}' = \begin{bmatrix} \left(\frac{\sigma_{mx}^2}{\sigma_{mx}^2+\sigma_x^2}\right)\sigma_x^2 & \left(\frac{\sigma_{mx}^2}{\sigma_{mx}^2+\sigma_x^2}\right)\sigma_{xx'}^2 & 0 & 0 & 0 \\ \left(\frac{\sigma_{mx}^2}{\sigma_{mx}^2+\sigma_x^2}\right)\sigma_{xx'}^2 & \sigma_{x'}^2 - \frac{\sigma_{xx'}^4}{\sigma_{mx}^2+\sigma_x^2} & 0 & 0 & 0 \\ 0 & 0 & \left(\frac{\sigma_{my}^2}{\sigma_{my}^2+\sigma_y^2}\right)\sigma_y^2 & \left(\frac{\sigma_{my}^2}{\sigma_{my}^2+\sigma_y^2}\right)\sigma_{yy'}^2 & 0 \\ 0 & 0 & \left(\frac{\sigma_{my}^2}{\sigma_{my}^2+\sigma_y^2}\right)\sigma_{yy'}^2 & \sigma_{y'}^2 - \frac{\sigma_{yy'}^4}{\sigma_{my}^2+\sigma_y^2} & 0 \\ 0 & 0 & 0 & 0 & \sigma_{p^{-1}}^2 \end{bmatrix}. \tag{22}$$

By way of interpretation, we can make the following conclusions regarding the filtered track parameters and errors.

- The filtered position and error are simply the weighted average of the prediction and measurement, a result which could have been easily arrived at without the Kalman fit.
- The filtered slope is updated to the extent that the slope is correlated with the position in the original track error matrix.
- The error of the filtered slope is reduced compared to the prediction provided that the correlation of the slope and position is positive.
- The track error matrix retains a 2+2+1 block diagonal structure (under our assumption that the space point errors are not correlated between x and y).
- The momentum and its error are are not modified by the Kalman fit under the assumptions we have made (zero correlation between position and momentum, which follows from zero magnetic field).

4 Vertical Wire Hit as Measurement

In this section, we assume that a measurement consists of a time t_m with error σ_t measured using a vertical wire (parallel to the y-axis). In other words, the measurement surface and track surface are both planes of constant z, which we take as the same surface. The predicted time depends on the track parameter x as follows,

$$t = x/v, \tag{23}$$

where v is the drift velocity. Following the development of the previous section, we define the residual r as,

$$r = t_m - t, \tag{24}$$

with error

$$\mathsf{R} = \sigma_r^2 = \sigma_t^2 + \sigma_x^2/v^2. \tag{25}$$

The Jacobian H of the prediction function is

$$\begin{aligned} \mathsf{H} &= \frac{\partial t}{\partial\left(x, x', y, y', p^{-1}\right)}, && (26) \\ &= \begin{bmatrix} \frac{\partial t}{\partial x} & \frac{\partial t}{\partial x'} & \frac{\partial t}{\partial y} & \frac{\partial t}{\partial y'} & \frac{\partial t}{\partial (p^{-1})} \end{bmatrix}, && (27) \\ &= \begin{bmatrix} \frac{1}{v} & 0 & 0 & 0 & 0 \end{bmatrix}. && (28) \end{aligned}$$

The Kalman gain matrix K is

$$\begin{aligned} \mathsf{K} &= \mathsf{C}\mathsf{H}^T\mathsf{R}^{-1}, && (29) \\ &= \begin{bmatrix} \sigma_x^2 & \sigma_{xx'}^2 & 0 & 0 & 0 \\ \sigma_{xx'}^2 & \sigma_{x'}^2 & 0 & 0 & 0 \\ 0 & 0 & \sigma_y^2 & \sigma_{yy'}^2 & 0 \\ 0 & 0 & \sigma_{yy'}^2 & \sigma_{y'}^2 & 0 \\ 0 & 0 & 0 & 0 & \sigma_{p^{-1}}^2 \end{bmatrix} \begin{bmatrix} \frac{1}{v} \\ 0 \\ 0 \\ 0 \\ 0 \end{bmatrix} \left(\frac{v^2}{v^2\sigma_t^2 + \sigma_x^2} \right), && (30) \\ &= \begin{bmatrix} \frac{v\sigma_x^2}{v^2\sigma_t^2+\sigma_x^2} \\ \frac{v\sigma_{xx'}^2}{v^2\sigma_t^2+\sigma_x^2} \\ 0 \\ 0 \\ 0 \end{bmatrix}. && (31) \end{aligned}$$

The filtered track parameter state vector is

$$\mathbf{x}' = \begin{bmatrix} \frac{xv^2\sigma_t^2 + t_m v \sigma_x^2}{v^2\sigma_t^2+\sigma_x^2} \\ x' + \left(\frac{v\sigma_{xx'}^2}{v^2\sigma_t^2+\sigma_x^2}\right)(t_m - t) \\ y \\ y' \\ 1/p \end{bmatrix}. \tag{32}$$

The filtered track error matrix is

$$\mathsf{C}' = \begin{bmatrix} \left(\frac{v^2\sigma_t^2}{v^2\sigma_t^2+\sigma_x^2}\right)\sigma_x^2 & \left(\frac{v^2\sigma_t^2}{v^2\sigma_t^2+\sigma_x^2}\right)\sigma_{xx'}^2 & 0 & 0 & 0 \\ \left(\frac{v^2\sigma_t^2}{v^2\sigma_t^2+\sigma_x^2}\right)\sigma_{xx'}^2 & \sigma_{x'}^2 - \frac{\sigma_{xx'}^4}{v^2\sigma_t^2+\sigma_x^2} & 0 & 0 & 0 \\ 0 & 0 & \sigma_y^2 & \sigma_{yy'}^2 & 0 \\ 0 & 0 & \sigma_{yy'}^2 & \sigma_{y'}^2 & 0 \\ 0 & 0 & 0 & 0 & \sigma_{p^{-1}}^2 \end{bmatrix}. \tag{33}$$

The results for this type of measurement are pretty similar to the case of the space point, except for the change of units between the track parameter x and time measurement, and the fact that the y track parameters are not affected.

5 Tilted Wire Hit as Measurement

In this section, we assume that a measurement consists of a time t_m with error σ_t measured using a wire rotated from vertical about the x-axis by angle θ. Assume that the track is specified on the surface $z = z_0$, and that the track surface intersects with the measurement surface along the line $y = y_0$, $z = z_0$ (see Fig. 1). The track intersects with the measurement surface at z-coordinate

$$z_m = z_0 + \frac{y - y_0}{\cot\theta - y'}. \tag{34}$$

The prediction function is

$$t = \frac{x}{v} + \frac{x'\,(y - y_0)}{v\,(\cot\theta - y')}. \tag{35}$$

The non-vanishing components of the Jacobian matrix (evaluated at $y = y_0$) are

$$\frac{\partial t}{\partial x} = \frac{1}{v}, \tag{36}$$

$$\frac{\partial t}{\partial y} = \frac{x'}{v\,(\cot\theta - y')}. \tag{37}$$

So, the full H matrix is

$$\mathsf{H} = \left[\begin{array}{ccccc} \frac{1}{v} & 0 & \frac{x'}{v(\cot\theta - y')} & 0 & 0 \end{array}\right]. \tag{38}$$

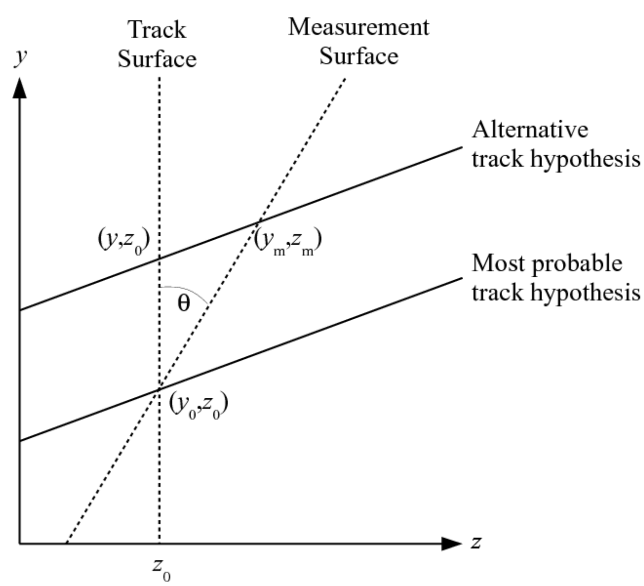

Figure 1: Intersection of track measured on z-plane with tilted wire surface, showing most probably track hypothesis, and alternative track hypothesis obtained by varying track parameter y.

The residual error is

$$\mathsf{R} = \sigma_r^2 = \sigma_t^2 + \frac{\sigma_x^2}{v^2} + \frac{x'^2 \sigma_y^2}{v^2 \left(\cot\theta - y'\right)^2}. \tag{39}$$

The Kalman gain matrix K is

$$\begin{aligned} \mathsf{K} &= \mathsf{C}\mathsf{H}^T\mathsf{R}^{-1}, & (40) \\ &= \begin{bmatrix} \sigma_x^2 & \sigma_{xx'}^2 & 0 & 0 & 0 \\ \sigma_{xx'}^2 & \sigma_{x'}^2 & 0 & 0 & 0 \\ 0 & 0 & \sigma_y^2 & \sigma_{yy'}^2 & 0 \\ 0 & 0 & \sigma_{yy'}^2 & \sigma_{y'}^2 & 0 \\ 0 & 0 & 0 & 0 & \sigma_{p^{-1}}^2 \end{bmatrix} \begin{bmatrix} \frac{1}{v} \\ 0 \\ \frac{x'}{v(\cot\theta - y')} \\ 0 \\ 0 \end{bmatrix} \left(\frac{1}{\sigma_r^2}\right), & (41) \\ &= \begin{bmatrix} \frac{\sigma_x^2}{v\sigma_r^2} \\ \frac{\sigma_{xx'}^2}{v\sigma_r^2} \\ \frac{x'\sigma_y^2}{v(\cot\theta - y')\sigma_r^2} \\ \frac{x'\sigma_{yy'}^2}{v(\cot\theta - y')\sigma_r^2} \\ 0 \end{bmatrix}. & (42) \end{aligned}$$

The filtered track parameter state vector is

$$\mathbf{x}' = \begin{bmatrix} x + \left(\frac{\sigma_x^2}{v\sigma_r^2}\right)(t_m - t) \\ x' + \left(\frac{\sigma_{xx'}^2}{v\sigma_r^2}\right)(t_m - t) \\ y + \left[\frac{x'\sigma_y^2}{v(\cot\theta - y')\sigma_r^2}\right](t_m - t) \\ y' + \left[\frac{x'\sigma_{yy'}^2}{v(\cot\theta - y')\sigma_r^2}\right](t_m - t) \\ 1/p \end{bmatrix}. \tag{43}$$

The filtered track error matrix is

$$\mathsf{C}' =$$

$$\begin{bmatrix} \sigma_x^2 - \frac{\sigma_x^4}{v^2\sigma_r^2} & \sigma_{xx'}^2 - \frac{\sigma_x^2\sigma_{xx'}^2}{v^2\sigma_r^2} & \frac{-x'\sigma_x^2\sigma_y^2}{v^2(\cot\theta - y')\sigma_r^2} & \frac{-x'\sigma_x^2\sigma_{yy'}^2}{v^2(\cot\theta - y')\sigma_r^2} & 0 \\ \sigma_{xx'}^2 - \frac{\sigma_x^2\sigma_{xx'}^2}{v^2\sigma_r^2} & \sigma_{x'}^2 - \frac{\sigma_{xx'}^4}{v^2\sigma_r^2} & \frac{-x'\sigma_{xx'}^2\sigma_y^2}{v^2(\cot\theta - y')\sigma_r^2} & \frac{-x'\sigma_{xx'}^2\sigma_{yy'}^2}{v^2(\cot\theta - y')\sigma_r^2} & 0 \\ \frac{-x'\sigma_x^2\sigma_y^2}{v^2(\cot\theta - y')\sigma_r^2} & \frac{-x'\sigma_{xx'}^2\sigma_y^2}{v^2(\cot\theta - y')\sigma_r^2} & \sigma_y^2 - \frac{x'^2\sigma_y^4}{v^2(\cot\theta - y')^2\sigma_r^2} & \sigma_{yy'}^2 - \frac{x'^2\sigma_y^2\sigma_{yy'}^2}{v^2(\cot\theta - y')^2\sigma_r^2} & 0 \\ \frac{-x'\sigma_x^2\sigma_{yy'}^2}{v^2(\cot\theta - y')\sigma_r^2} & \frac{-x'\sigma_{xx'}^2\sigma_{yy'}^2}{v^2(\cot\theta - y')\sigma_r^2} & \sigma_{yy'}^2 - \frac{x'^2\sigma_y^2\sigma_{yy'}^2}{v^2(\cot\theta - y')^2\sigma_r^2} & \sigma_{y'}^2 - \frac{x'^2\sigma_{yy'}^4}{v^2(\cot\theta - y')^2\sigma_r^2} & 0 \\ 0 & 0 & 0 & 0 & \sigma_{p^{-1}}^2 \end{bmatrix}. \tag{44}$$

Here are a couple of observations about the filtered track parameters and errors.

- In general, both x and y track parameters are modified, and their errors are reduced, by the tilted wire measurement.

- The y track parameters are not affected if $\theta = 0$ or if the track is traveling parallel to the wire plane ($x' = 0$). In either case, the track update is the same as for a vertical wire measurement.

- The x and y track parameters get correlated by the tilted wire measurement. However, if there are wires with both positive and negative θ, the average correlation will tend to cancel out over many measurements.

- The prediction function and updating formulas are singular if the track is traveling parallel to the wire in the yz-plane ($y' = \cot\theta$).

With respect to the last point, among other pathologies, observe that the filtered error σ_y^2 has the curious property that this error is exactly zero if $y' = \cot\theta$. This actually makes a kind of sense if you assume that the track surface is known exactly (by assumption) and the measurement surface is also known exactly, and the track exactly coincides with the measurement surface. What is missing in this analysis is the fact that the measurement surface actually has a nonzero thickness. The thickness of the measurement surface contributes to the time measurement error σ_t. In practice, a track that was traveling parallel or nearly parallel to a wire should produce a very broad pulse, and therefore, perhaps, a large intrinsic time measurement error σ_t. The divergence of σ_t might prevent the filtered σ_y from becoming too small in cases where a track is nearly parallel to a wire. Another way the updating formula breaks down near $y' = \cot\theta$ is the assumption of linearity of the prediction function over a range allowed by the track parameter errors. For these reasons, one should be alert for possible pathologies in the case of a track traveling parallel to a wire in the yz plane.